A taste of Girard’s Transcendental Syntax

Team LoVe – LIPN Université Sorbone Paris Nord

Boris ENG & Thomas Seiller

Transcendental Syntax

Geometry of Interaction : proof-nets from the mathematics of cut-elimination

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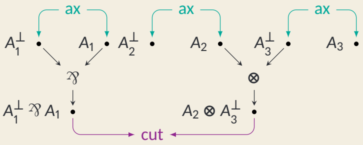

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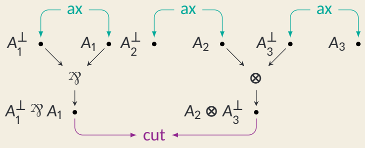

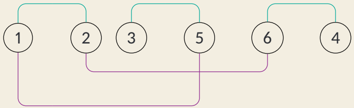

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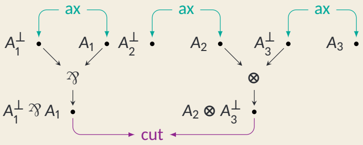

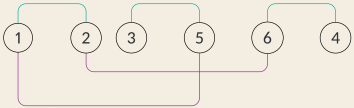

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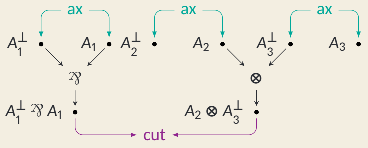

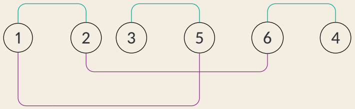

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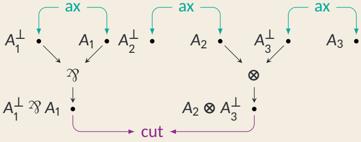

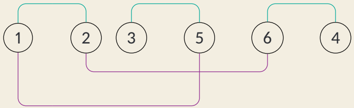

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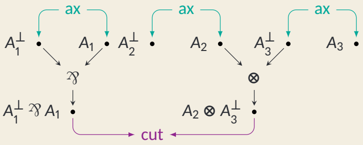

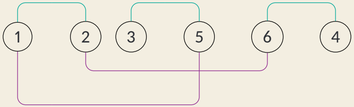

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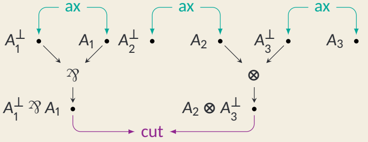

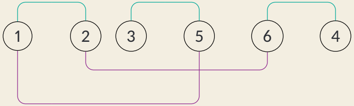

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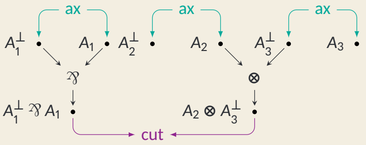

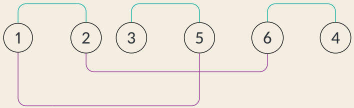

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- GoI 6 : extension of this approach
- Transcendental Syntax : same but with different name and motivations.

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Term unification

First-order terms. $t, u ::= x \mid f(t_1, \ldots, t_n)$

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↳ for $x \doteq f(x) \simeq_\alpha y \doteq f(x)$ we have $\theta = y \longmapsto f(x)$

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Stars and constellations

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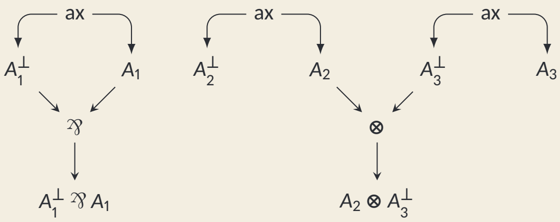
Unlike logic programming : no logic/meaning, no contradiction $\bot$, no goal/query.

Multiplicative Linear Logic

Interpreting the dynamics of proofs



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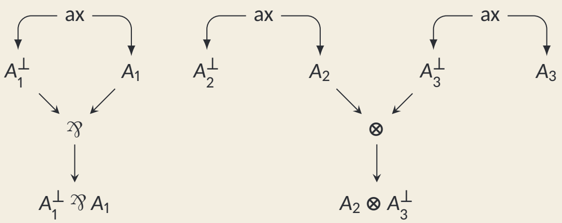

$p_{A_1^\perp ⅋ A_1}(\mathtt{l} \cdot x)$	$p_{A_1^\perp ⅋ A_1}(\mathtt{r} \cdot x)$

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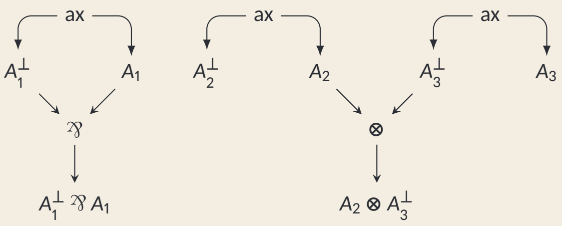

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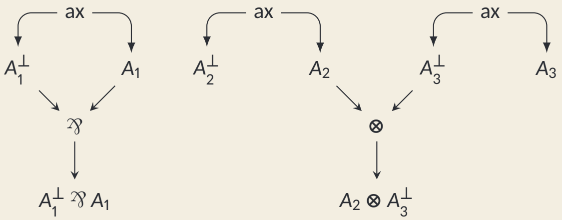

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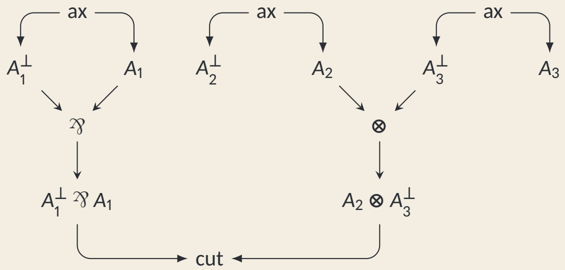

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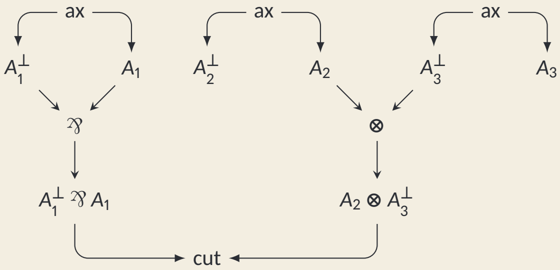

$+c.p_{A_1^\perp ⅋ A_1}(\mathtt{l}\cdot x)$	$+c.p_{A_1^\perp ⅋ A_1}(\mathtt{r}\cdot x)$

$p_{A_2^\perp}(x)$	$+c.p_{A_2\otimes A_3^\perp}(\mathtt{l}\cdot x)$

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$-c.p_{A_1^\perp ⅋ A_1}(x)$	$-c.p_{A_2\otimes A_3^\perp}(x)$

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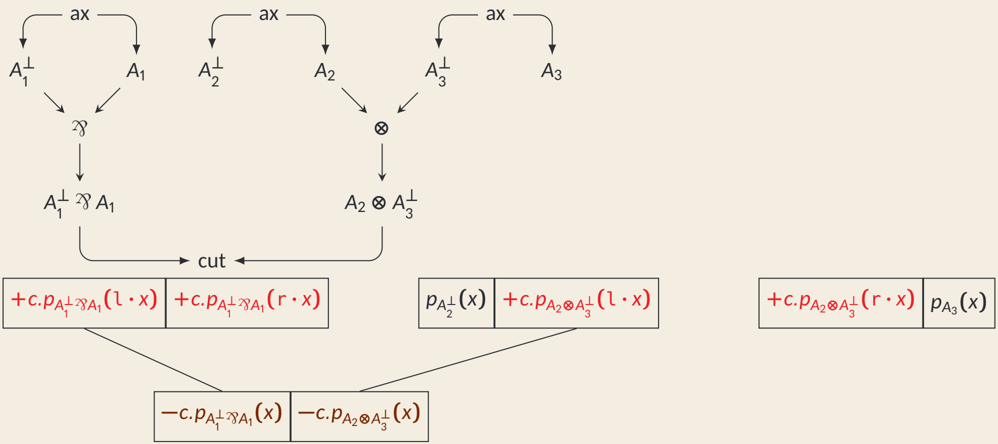

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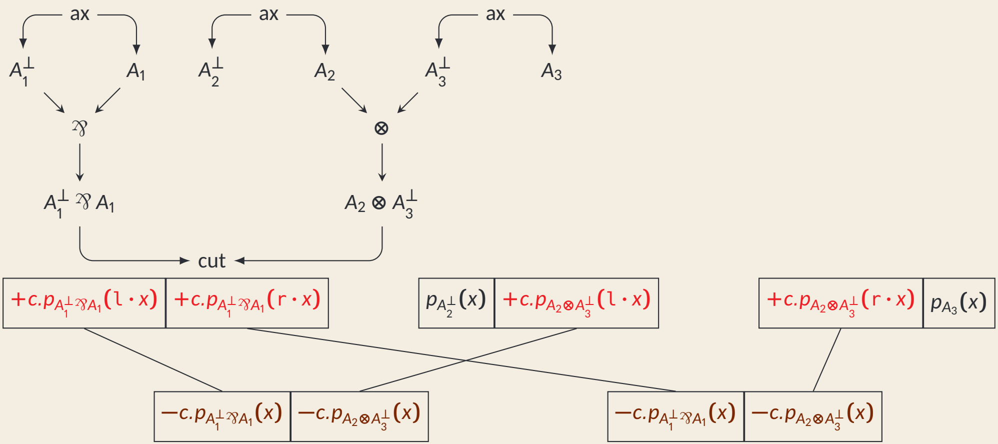

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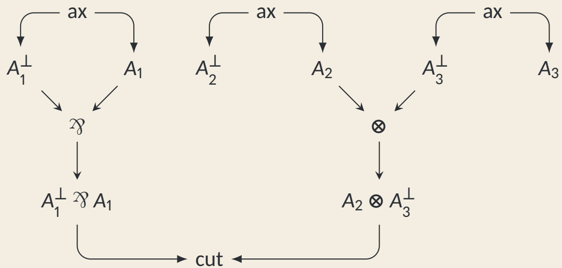

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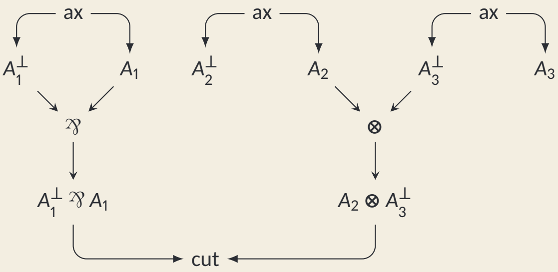

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Cut-elimination : resolution of contraints on addresses

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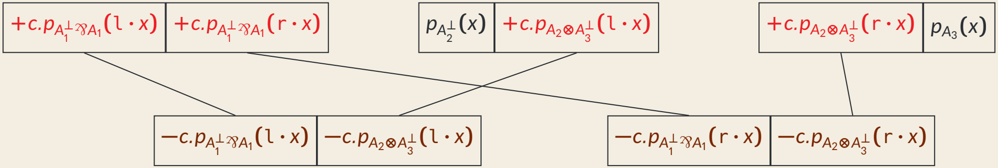
Liberalisation of proofs



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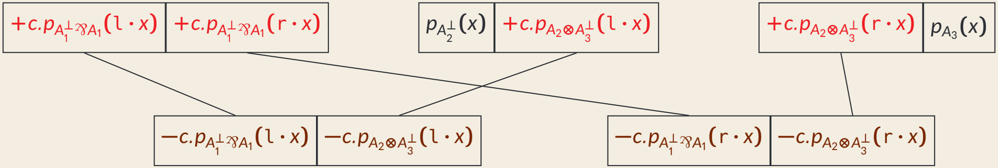

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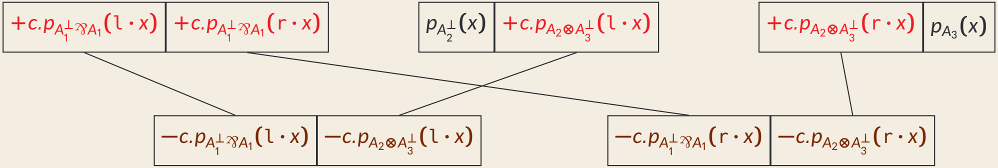

- pre-proof of $\vdash A$ $\{[p_A(x)]\}$
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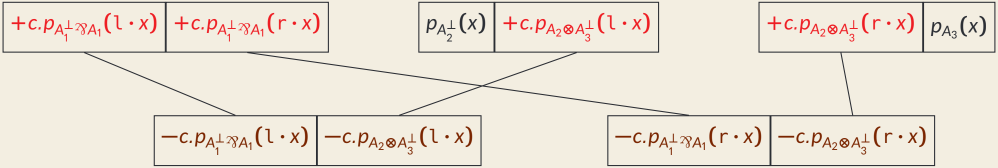

- pre-proof of $\vdash A$ $\{[p_A(x)]\}$
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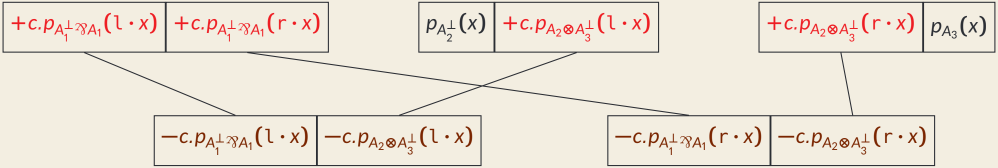

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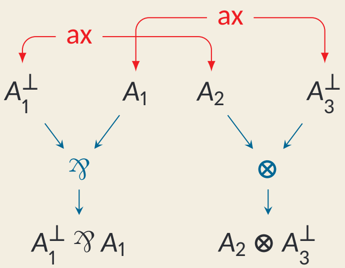
Generalises permutations but also partitions [Acclavio, Maieli]

Multiplicative Linear Logic

Girard's factory : vehicle and tests



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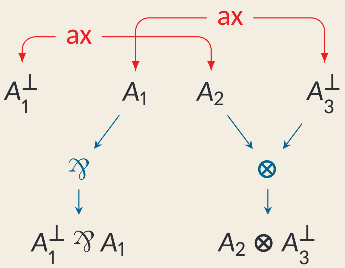

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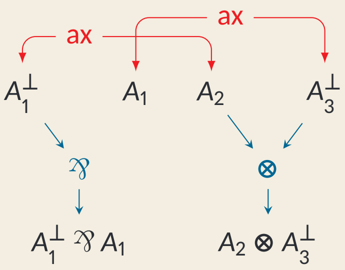

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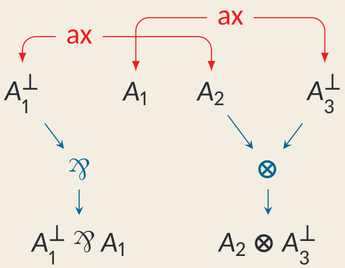

Danos-Regnier correctness $\longrightarrow$ Vehicle + Test = certification of proof-net

Multiplicative Linear Logic

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$+t.p_{A\otimes B}(\mathtt{l}\cdot x)$	$+t.p_{A^{\perp}⅋B^{\perp}}(\mathtt{l}\cdot x)$

$+t.p_{A\otimes B}(\mathtt{r}\cdot x)$	$+t.p_{A^{\perp}⅋B^{\perp}}(\mathtt{r}\cdot x)$

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$\boxed{+t.p_{A\otimes B}(l\cdot x)}\boxed{+t.p_{A^\perp \parr B^\perp}(l\cdot x)}$ $\boxed{+t.p_{A\otimes B}(r\cdot x)}\boxed{+t.p_{A^\perp \parr B^\perp}(r\cdot x)}$

$\boxed{\left[\frac{-t.p_{A\otimes B}(l\cdot x)}{+c.q_A(x)}\right]}$ $\boxed{\left[\frac{-t.p_{A\otimes B}(r\cdot x)}{+c.q_{A^\perp}(x)}\right]}$ $\boxed{\left[\frac{-t.p_{A^\perp \parr B^\perp}(l\cdot x)}{+c.q_B(x)}\right]}$ $\boxed{\left[\frac{-t.p_{A^\perp \parr B^\perp}(r\cdot x)}{+c.q_{B^\perp}(x)}\right]}$

$\boxed{\left[\frac{-c.q_A(x) \quad -c.q_B(x)}{+c.q_{A\otimes B}(x)}\right]}$ $\boxed{\frac{-c.q_{A^\perp}(x)}{}}$ $\boxed{\frac{-c.q_{B^\perp}(x)}{+c.q_{A^\perp \parr B^\perp}(x)}}$

$\boxed{\left[\frac{-c.q_{A\otimes B}(x)}{p_{A\otimes B}(x)}\right]}$ $\boxed{\left[\frac{-c.q_{A^\perp \parr B^\perp}(x)}{p_{A^\perp \parr B^\perp}(x)}\right]}$

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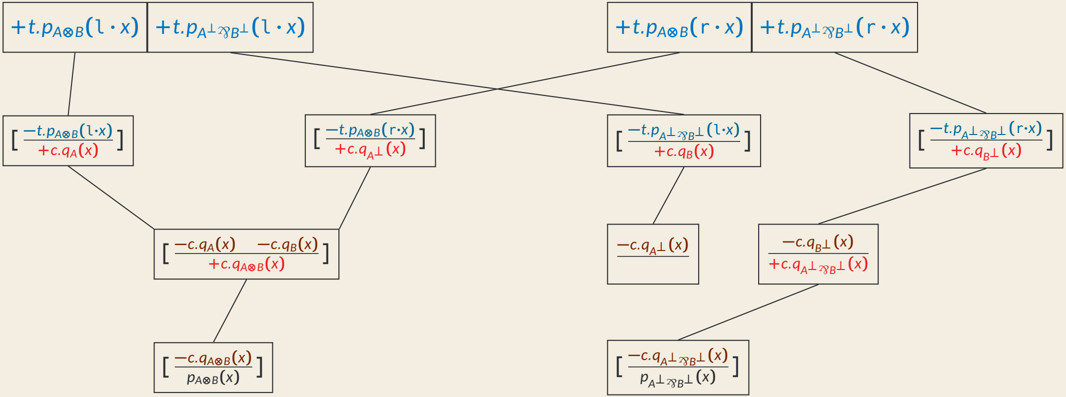

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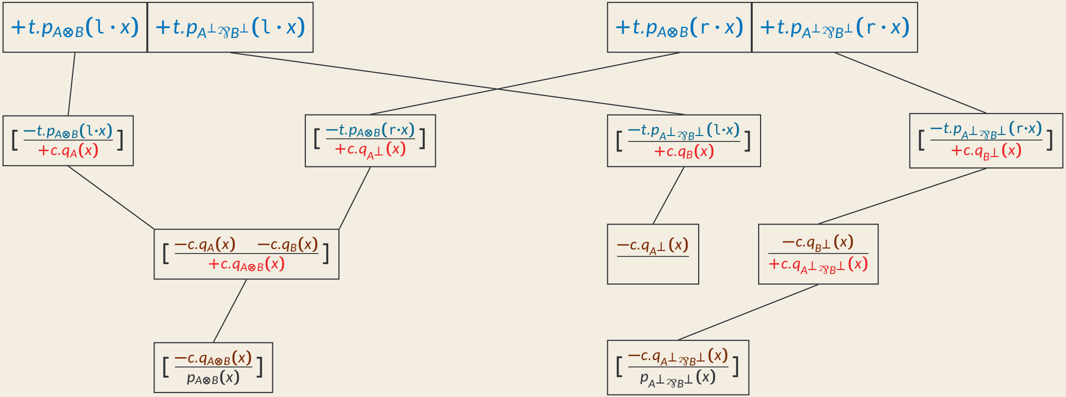

correct iff for all test Φ_T we have $\mathrm{Ex}(\Phi_V \uplus \Phi_T) = [p_{A_1}(x), \ldots, p_{A_n}(x)]$.

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Testing and typing

Similar to testing in programming but with finitely many tests. $\Phi, \Phi' : \mathrm{Ex}(\Phi \uplus \Phi')$?

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- $\Phi \perp \Phi'$ when $|\mathrm{Ex}(\Phi \uplus \Phi')| < \infty$: MLL+MIX (acyclic tests).

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- $\Phi \perp \Phi'$ when $|\mathrm{Ex}(\Phi \uplus \Phi')| < \infty$: MLL+MIX (acyclic tests).
- $\Phi \perp \Phi'$ when $|\mathrm{Ex}(\Phi \uplus \Phi')| = 1$: MLL (acyclic and connected tests).

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