# A gentle introduction to Girard’s Transcendental Syntax

LIPN – Université Sorbonne Paris Nord

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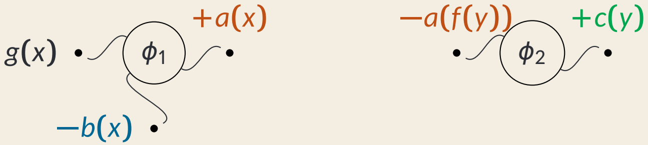
↳ **Computational bricks** : "stellar resolution" (not the only possibility).

↳ **Logical correctness** : by symmetric computational testing.

# Stellar Resolution

*Between tilings and logic programming*

"Flexible" tiles [stars] with (un)polarised terms [rays]. Group as [constellations].



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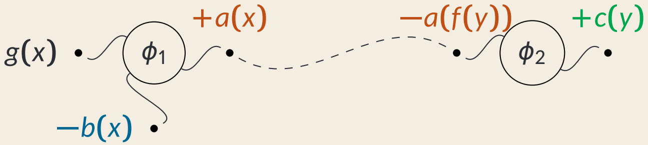

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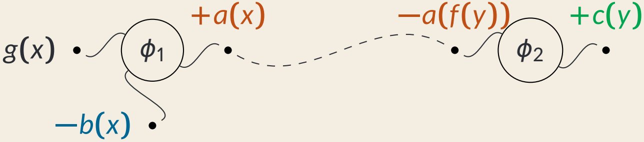


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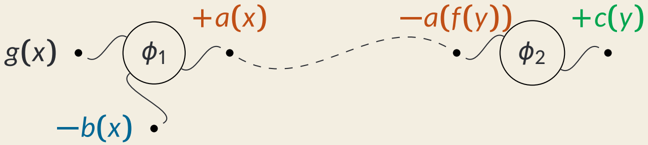


**Evaluation** : link-contraction by Robinson's Resolution rule.

**Execution** : construct all possible connected & maximal tilings then evaluate them.

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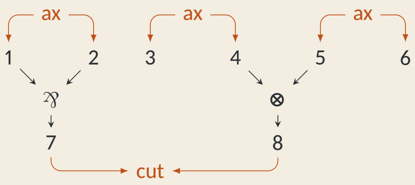

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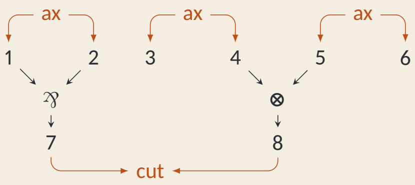


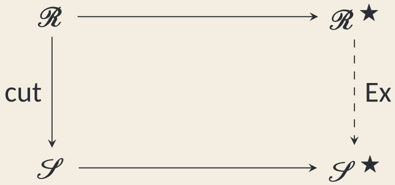

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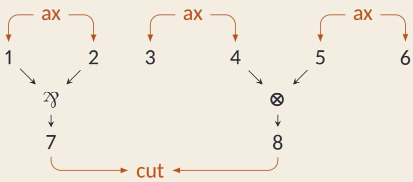


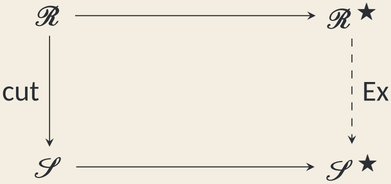


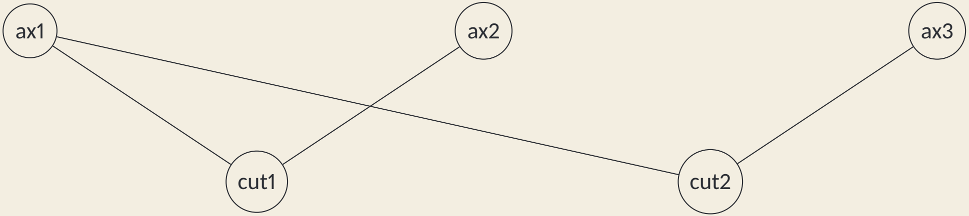

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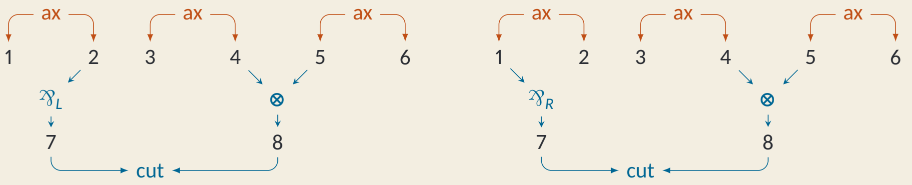

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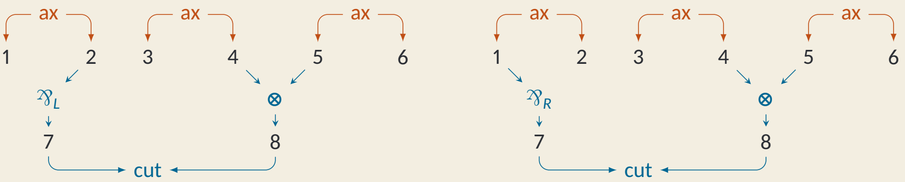


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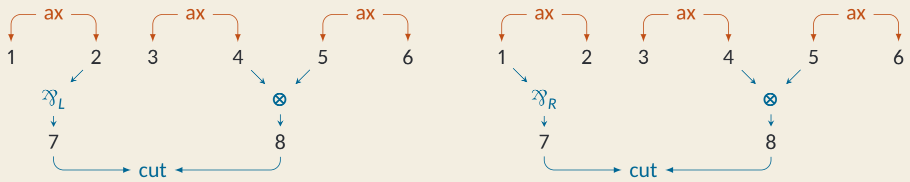


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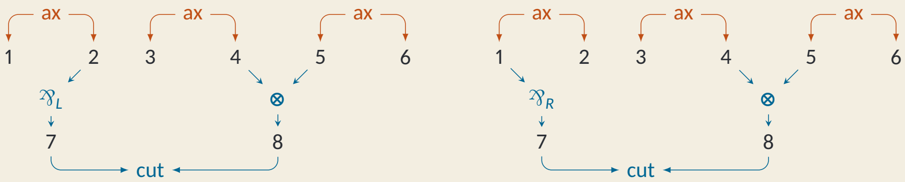


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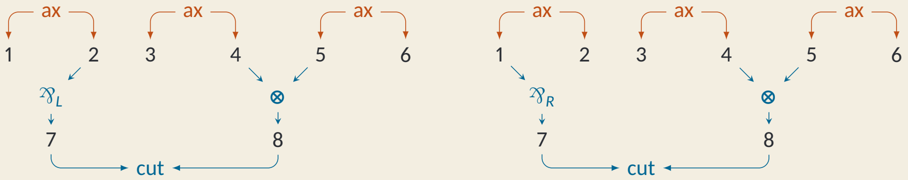


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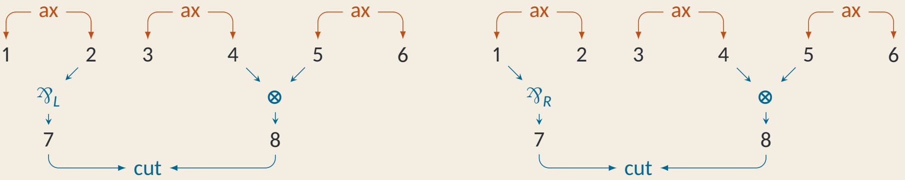


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**Orthogonality.** $\mathrm{Ex}(\Phi_1 \uplus \Phi_2)$ satisfies $P \iff \Phi_1 \perp \Phi_2$.

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**Types as labels (type theory).** $A, B ::= X_i \mid X_i^\perp \mid A \otimes B \mid A \parr B$.

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Related by adequacy : $\texttt{Tests}(A)^\perp \subseteq \mathbf{A}$.

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